# ON THE RECURSIVE STRUCTURE OF BRANSON'S
# $Q$-CURVATURE

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ABSTRACT. We prove universal recursive formulas for Branson's $Q$-curvatures in terms of respective lower-order $Q$-curvatures, lower-order GJMS-operators and holographic coefficients.

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## CONTENTS

## 1. INTRODUCTION AND FORMULATION OF THE MAIN RESULT

On any Riemannian manifold $(M, g)$ of *even* dimension $n$, there is a finite sequence $P_2(g), P_4(g), \ldots, P_n(g)$ of geometric differential operators of the form[1]

$$\Delta_g^N + \text{lower order terms}$$

which are conformally covariant in the sense that

$$e^{(\frac{n}{2}+N)\varphi} P_{2N}(e^{2\varphi}g)(u) = P_{2N}(g)(e^{(\frac{n}{2}-N)\varphi}u) \tag{1.1}$$

for all $\varphi \in C^\infty(M)$. Similarly, on manifolds of *odd* dimension there is an infinite sequence $P_2(g), P_4(g), \ldots$ of geometric operators satisfying (1.1). The operators $P_{2N}(g)$ are *geometric* in the sense that the lower order terms are determined by the metric and its curvature. These operators were constructed in the seminal work [GJMS92]. They will be referred to as the GJMS-operators.

The constant terms of the GJMS-operators lead to the notion of Branson's $Q$-curvatures (see [B95]). In fact, for $2N < n$ it is natural to write the constant term of $P_{2N}$ in the form

$$P_{2N}(g)(1) = (-1)^N \left( \frac{n}{2} - N \right) Q_{2N}(g) \tag{1.2}$$

[1]We use the convention that $-\Delta \geq 0$

with a scalar Riemannian curvature invariant $Q_{2N}(g) \in C^\infty(M)$ of order $2N$. For even $n$, the *critical* GJMS-operator $P_n$ has vanishing constant term and (1.2) cannot be used to define an analogous quantity $Q_n$. However, $Q_n$ can be defined through $Q_{2N}$ for $2N < n$ by a continuation in the dimension. The quantities $Q_{2N}$ will be called Branson's $Q$-curvatures. For even $n$, $Q_n$ will be called the *critical* $Q$-curvature.

The following two special cases are well-known. We have

$$Q_2 = \frac{\text{scal}}{2(n-1)}$$

and

$$Q_4 = \frac{n}{2}\mathsf{J}^2 - 2|\mathsf{P}|^2 - \Delta(\mathsf{J}), \ n \geq 3, \tag{1.3}$$

where we used the abbreviations

$$\mathsf{J} = \frac{\text{scal}}{2(n-1)} \quad \text{and} \quad \mathsf{P} = \frac{1}{n-2}(\text{Ric} - \mathsf{J}g).$$

$\mathsf{P}$ is the Schouten tensor. The quantities $Q_2$ and $Q_4$ appear in the corresponding Yamabe and Paneitz operators

$$P_2 = \Delta - \left(\frac{n}{2} - 1\right)Q_2$$

and

$$P_4 = \Delta^2 + \delta((n-2)\mathsf{J} - 4\mathsf{P})d + \left(\frac{n}{2} - 2\right)Q_4.$$

The main purpose of this paper is to establish formulas for all higher order $Q$-curvatures.

In order to formulate the main result, we need some more notation.

First, a sequence $I = (I_1, \ldots, I_r)$ of integers $I_j \geq 1$ will be regarded as a composition of the sum $|I| = I_1 + I_2 + \cdots + I_r$, where two representations which contain the same summands but differ in the order of the summands are regarded as different. $|I|$ will be called the size of $I$. For $I = (I_1, \ldots, I_r)$, we set

$$P_{2I} = P_{2I_1} \circ \cdots \circ P_{2I_r}.$$

We define the multiplicity $m_I$ of the composition $I$ by

$$m_I = -(-1)^r |I|! \, (|I| - 1)! \prod_{j=1}^{r} \frac{1}{I_j! \, (I_j - 1)!} \prod_{j=1}^{r-1} \frac{1}{I_j + I_{j+1}}. \tag{1.4}$$

Here, an empty product has to be interpreted as 1. Note that $m_{(N)} = 1$ for all $N \geq 1$ and

$$\sum_{|I|=N} m_I = 0$$

(see Lemma 2.1 in [J09b]). In these terms, we introduce the generating function

$$\mathcal{G}(r) = 1 + \sum_{N \geq 1} \left( \sum_{a+|J|=N} m_{(J,a)}(-1)^a P_{2J}(Q_{2a}) \right) \frac{r^N}{N!(N-1)!}. \tag{1.5}$$

Here, for even $n$, the sum on the right-hand side is to be understood as a finite sum over $1 \leq N \leq n$.

A second ingredient of our formula for $Q$-curvatures comes from Poincaré-Einstein metrics. Let $n$ be even. For a given metric $g$ on the manifold $M$ of dimension $n$, let

$$g_+ = r^{-2}(dr^2 + g_r) \tag{1.6}$$

with

$$g_r = g + r^2 g_{(2)} + \cdots + r^{n-2} g_{(n-2)} + r^n (g_{(n)} + \log r \bar{g}_{(n)}) + \cdots \tag{1.7}$$

be a metric on $X = (0, \varepsilon) \times M$ so that the tensor $\mathrm{Ric}(g_+) + n g_+$ satisfies the Einstein condition

$$\mathrm{Ric}(g_+) + n g_+ = O(r^{n-2}) \tag{1.8}$$

together with a certain vanishing trace condition. These conditions uniquely determine the coefficients $g_{(2)}, \ldots, g_{(n-2)}$ and the quantity $\mathrm{tr}_g(g_{(n)})$. They are given as polynomial formulas in terms of $g$, its inverse, the curvature tensor of $g$, and its covariant derivatives. A metric $g_+$ with these properties is called a Poincaré-Einstein metric with conformal infinity $[g]$. Similarly, for odd $n$, the Einstein condition determines all coefficients in the formal power series

$$g_r = g + r^2 g_{(2)} + r^4 g_{(4)} + \cdots$$

with only even powers of $r$. For full details see [FG07]. The volume form of $g_+$ can be written as

$$vol(g_+) = r^{-n-1} v(r) dr vol(g),$$

where

$$v(r) = vol(g_r)/vol(g) \in C^\infty(M).$$

The coefficients in the Taylor series

$$v(r) = v_0 + v_2 r^2 + v_4 r^4 + \cdots$$

are known as the *renormalized volume* coefficients [G00], [G09] or *holographic* coefficients [J09a]. The coefficient $v_{2j} \in C^\infty(M)$ is given by a local formula which involves at most $2j$ derivatives of the metric.

The following theorem is the main result of the present paper. It settles Conjecture 9.2 in [J09b].

**Theorem 1.1.** *On any Riemannian manifold $M$ of dimension $n \geq 3$,*

$$\mathcal{G}\left(\frac{r^2}{4}\right) = \sqrt{v(r)}. \tag{1.9}$$

Some comments are in order. The relation (1.9) is to be understood as an identity of formal power series in $r$. Moreover, for even $n$, it is to be understood as an identity of finite power series terminating at $r^n$. The Taylor coefficients of

$$w(r) = \sqrt{v(r)} = 1 + w_2 r^2 + w_4 r^4 + \cdots$$

can be expressed in terms of the holographic coefficients $v_2, v_4, \cdots$. In particular, we have

$$2w_2 = v_2,$$
$$8w_4 = 4v_4 - v_2^2,$$
$$16w_6 = 8v_6 - 4v_4v_2 + v_2^3,$$
$$128w_8 = 64v_8 - 32v_6v_2 - 16v_4^2 + 24v_2^2v_4 - 5v_2^4.$$

Note that

$$-2v_2 = \mathsf{J} \quad \text{and} \quad 8v_4 = \mathsf{J}^2 - |\mathsf{P}|^2. \tag{1.10}$$

Graham [G09] describes an algorithm to derive formulas for holographic coefficients in terms of the metric, and displays explicit formulas for $v_6$ and $v_8$. In particular,

$$-48v_6 = 6\,\mathrm{tr}(\wedge^3\mathsf{P}) - 2(\Omega^{(1)}, \mathsf{P}),$$

where $\Omega^{(1)}$ denote Graham's first extended obstruction tensor.[2] We refer to [J09a] for the details of such calculations. Note also that for locally conformally flat metrics,

$$(-2)^N v_{2N} = \mathrm{tr}(\wedge^N\mathsf{P}). \tag{1.11}$$

Theorem 1.1 provides recursive formulas for $Q_{2N}$ in the following way. For any $N \geq 1$ (so that $2N \leq n$ if $n$ is even), (1.9) states that

$$\sum_{a+|J|=N} m_{(J,a)}(-1)^a P_{2J}(Q_{2a}) = 2^{2N}N!(N-1)!w_{2N}. \tag{1.12}$$

One of the $2^{N-1}$ items in the sum on the left-hand side of (1.12) is $(-1)^N Q_{2N}$. All other items are defined in terms of lower-order GJMS-operators acting on lower-order $Q$-curvatures.

The relation (1.12) can be regarded as a formula for the difference

$$Q_{2N} - (-1)^N 2^{2N-1}N!(N-1)!v_{2N}.$$

An alternative formula for the same difference is given by the *holographic* formula

$$2Nc_N Q_{2N} = \sum_{j=0}^{N-1}(N-j)\mathcal{T}_{2j}^*\left(\frac{n}{2}-N\right)(v_{2N-2j});$$

for the notation see Section 3. In the critical case $2N = n$, the latter formula was proved in [GJ07]. For the general case we refer to [J10a].

The identities (1.9) are valid in *all* dimensions. This feature will be referred to as *universality*.

For the convenience of the reader, we display the explicit formulas for the four lowest order $Q$-curvatures. We find

$$Q_2 = -4w_2,$$
$$Q_4 = -P_2(Q_2) + 2^4 2! w_4,$$

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[2]In more familiar terms, $\Omega^{(1)}$ equals $\mathcal{B}/4-n$, where $\mathcal{B}$ is a version of the Bach tensor in dimension $n$.

$$Q_6 = -2P_2(Q_4) + 2P_4(Q_2) - 3P_2^2(Q_2) - 2^6 3! 2! w_6$$

and

$$\begin{aligned} Q_8 = &-3P_2(Q_6) - 3P_6(Q_2) + 9P_4(Q_4) \\ &+ 8P_2 P_4(Q_2) - 12 P_2^2(Q_4) + 12 P_4 P_2(Q_2) - 18 P_2^3(Q_2) + 2^8 4! 3! w_8. \quad (1.13) \end{aligned}$$

By $P_2 = \Delta - (\frac{n}{2} - 1)\mathsf{J}$ and (1.10), the formula for $Q_4$ is easily seen to be equivalent to (1.3). Similarly, combining formula (1.13) with recursive formulas for $P_4$ and $P_6$ yields more explicit presentations of $Q_8$. For the details concerning such consequences we refer to [J09d].

For round spheres $\mathbb{S}^n$, Theorem 1.1 was proved in [J09b] by direct summation of the left-hand side. The expectation that Theorem 1.1 holds true also for pseudo-Riemannian metrics is supported by the summation formulas on $\mathbb{S}^{p,q}$ proved in [J09c].

Finally, we emphasize that the variable $r$ plays different roles on both sides of (1.9). In fact, $r$ is used as a formal variable of a generating function on the left-hand side and as a defining function on the right-hand side.

The paper is organized as follows. In Section 2, we establish explicit formulas for a sequence of recursively defined operator-valued polynomials $\pi_{2N}(\lambda)$. These are closely related to the $Q$-curvature polynomials $Q_{2N}^{res}(\lambda)$ that were introduced in [J09a]. In Section 3, we recall this concept and show that the relation implies Theorem 1.1, when combined with a result of [J09d]. In Section 4, we prove a summation formula for GJMS-operators on round spheres which is parallel to a summation formula for GJMS-operators proved in [J09b].

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## 2. THE POLYNOMIALS $\pi_{2N}(\lambda)$

In the present section, we discuss a sequence of operator-valued polynomials which are closely related to the $Q$-curvature polynomials. That relation will be important in Section 3.

We start by defining some higher analogs of the multiplicities $m_I$. We set $m_I^{(1)} = m_I$, and define the rational numbers $m_I^{(k)}$ for $k \geq 2$ by the formulas

$$m_{(a,J)}^{(k)} = \frac{\sum_{j=0}^{k-1} s(N, N-j)|J|^{k-1-j}}{(N-1)\cdots(N-k+1)} m_{(a,J)}^{(1)} \quad (2.1)$$

if $a + |J| = N$ and $2 \leq k \leq N - 1$, and

$$m_{(N)}^{(k)} = \frac{s(N, N-k+1)}{(N-1)\cdots(N-k+1)} m_{(N)}^{(1)} \quad (2.2)$$

for $2 \leq k \leq N$. Note that (2.2) (for $2 \leq k \leq N - 1$) can be regarded as the special case $J = (0)$ of (2.1).

Here, $s(n, m)$ are the Stirling numbers of the first kind. These are defined by the generating functions

$$\sum_{k=0}^{n} s(n, k) x^k = x(x-1) \cdots (x-n+1) = b_n(x). \tag{2.3}$$

In particular, we have

$$s(n, 1) = (-1)^{n-1}(n-1)!, \quad s(n, n-1) = -\binom{n}{2} \quad \text{and} \quad s(n, n) = 1. \tag{2.4}$$

Note that the definitions show that

$$m_{(a,J)}^{(2)} = \frac{s(N, N)|J| + s(N, N-1)}{N-1} m_{(a,J)}^{(1)} = \left(\frac{N-a}{N-1} - \frac{N}{2}\right) m_{(a,J)}^{(1)} \tag{2.5}$$

if $a + |J| = N$, and

$$m_{(N)}^{(2)} = \frac{s(N, N-1)}{N-1} = -\frac{N}{2} m_{(N)}^{(1)}.$$

Finally, we use the operators

$$\mathbf{C}_{2N}^{(k)} = \sum_{|I|=N} m_I^{(k)} P_{2I} \quad \text{for } 1 \le k \le N-1 \tag{2.6}$$

and

$$\mathbf{C}_{2N}^{(N)} = (-1)^{N-1} P_{2N} \tag{2.7}$$

to define the operator-valued polynomials

$$\pi_{2N}(\lambda) = \sum_{k=1}^{N} \mathbf{C}_{2N}^{(k)} \frac{1}{(N-k)!} \left(\lambda + \frac{n}{2} - N\right)^{N-k}, \ N \ge 1. \tag{2.8}$$

We display explicit formulas for these polynomials for $N \le 3$.

**Examples 2.1.** *We have,* $\pi_2(\lambda) = P_2$,

$$\pi_4(\lambda) = (P_4 - P_2^2) \left(\lambda + \frac{n}{2} - 2\right) - P_4$$

*and*

$$\pi_6(\lambda) = (P_6 - 2P_4 P_2 - 2P_2 P_4 + 3P_2^3) \frac{1}{2!} \left(\lambda + \frac{n}{2} - 3\right)^2$$
$$+ \left(-\frac{3}{2} P_6 + 2P_4 P_2 + P_2 P_4 - \frac{3}{2} P_2^3\right) \left(\lambda + \frac{n}{2} - 3\right) + P_6.$$

Note that $m_I = m_{I^{-1}}$, where $I^{-1}$ is the inverse composition of $I$. Since the GJMS-operators are formally self-adjoint (see [GZ03]), this fact implies that $\mathbf{C}_{2N}^{(1)}$ is formally self-adjoint, too.

The main result of the present section consists in the following characterization of the polynomials $\pi_{2N}(\lambda)$.

**Theorem 2.1.** *For any $N \geq 1$, the polynomial $\pi_{2N}(\lambda)$ satisfies the $N$ identities*

$$\pi_{2N}\left(-\frac{n}{2}+2N-j\right) = (-1)^j P_{2j}\,\pi_{2N-2j}\left(-\frac{n}{2}+2N-j\right),\ j = 1, \ldots, N-1 \qquad (2.9)$$

*and*

$$\pi_{2N}\left(-\frac{n}{2}+N\right) = (-1)^{N-1}P_{2N}. \qquad (2.10)$$

Since $\pi_{2N}(\lambda)$ has degree $N-1$, the factorizations (2.9) and (2.10) uniquely determine this polynomial in terms of the lower-order relatives $\pi_2, \ldots, \pi_{2N-2}$ and the GJMS-operator $P_{2N}$.

As a preparation of the proof of Theorem 2.1 we need the following result.

**Lemma 2.1.** *For all $N \geq 2$,*

$$\sum_{2 \leq a+b \leq N} s(N, a+b)x^a y^b = \frac{yx(x-1)\cdots(x-N+1) - xy(y-1)\cdots(y-N+1)}{x-y} \qquad (2.11)$$

*for $x \neq y$. Moreover,*

$$\sum_{2 \leq a+b \leq N} s(N, a+b)M^{a+b-1} = (-1)^{N-M-1}M!(N-M-1)! \qquad (2.12)$$

*for $M = 0, 1, \ldots, N-1$. Here the sums run over all natural numbers $a, b \geq 1$ subject to the condition $2 \leq a + b \leq N$.*

*Proof.* The sum in (2.11) can be written in the form

$$s(N, N)\sum_{a=1}^{N-1} x^a y^{N-a} + s(N, N-1)\sum_{a=1}^{N-2} x^a y^{N-a-1} + \cdots + s(N, 2)xy. \qquad (2.13)$$

Now (2.13) equals

$$\sum_{k=2}^{N} s(N, k)\left(y\frac{x^k - y^k}{x-y} - y^k\right)$$

$$= \frac{y}{x-y}\left(\sum_{k=2}^{N} s(N, k)x^k - \sum_{k=2}^{N} s(N, k)y^k\right) - \sum_{k=2}^{N} s(N, k)y^k.$$

In view of (2.3), the latter sum simplifies to

$$\frac{y}{x-y}\left(x(x-1)\cdots(x-N+1) - y(y-1)\cdots(y-N+1) - s(N, 1)x + s(N, 1)y\right)$$

$$- (y(y-1)\cdots(y-N+1) - s(N, 1)y).$$

Now (2.11) follows from here by a further simplification. Finally, (2.12) follows from (2.11) by taking the limit $x \to y$ for $y = 0, 1, \ldots, N-1$. $\qquad \square$

We continue with the

*Proof of Theorem 2.1.* (2.10) is obvious by (2.7). The left-hand side of (2.9) equals

$$\sum_{k=1}^{N-1} \mathbf{C}_{2N}^{(k)} \frac{1}{(N-k)!}(N-j)^{N-k} + (-1)^{N-1}P_{2N}. \tag{2.14}$$

We prove that all non-trivial contributions to this sum are multiples of the operators

$$P_{2j}P_{2J}, \ j + |J| = N.$$

Moreover, we determine the corresponding weights. Let $J$ be non-trivial. Eq. (2.1) shows that the coefficient of $P_{2j}P_{2J}$ in

$$\sum_{k=1}^{N-1} \mathbf{C}_{2N}^{(k)} \frac{1}{(N-k)!}(N-j)^{N-k} \tag{2.15}$$

is given by

$$\sum_{k=1}^{N-1} \left( \frac{\sum_{i=0}^{k-1} s(N, N-i)(N-j)^{k-1-i}}{(N-1)\cdots(N-k+1)} \right) \frac{(N-j)^{N-k}}{(N-k)!} \, m_{(j,J)}^{(1)}$$

$$= \frac{1}{(N-1)!} \sum_{k=1}^{N-1}\sum_{i=0}^{k-1} s(N, N-i)(N-j)^{N-1-i} \, m_{(j,J)}^{(1)}$$

$$= \frac{1}{(N-1)!} \sum_{2 \leq i+k \leq N} s(N, i+k)(N-j)^{i+k-1} \, m_{(j,J)}^{(1)}.$$

Eq. (2.12) implies that the latter sum equals

$$(-1)^{j-1}\frac{(N-j)!(j-1)!}{(N-1)!}m_{(j,J)}^{(1)} = (-1)^{j-1}\binom{N-1}{j-1}^{-1} m_{(j,J)}^{(1)} \tag{2.16}$$

for $1 \leq j \leq N-1$.

Next, (2.3) shows that $P_{2N}$ contributes to (2.13) with the coefficient

$$(-1)^{N-1} + \sum_{k=1}^{N-1} m_{(N)}^{(k)} \frac{1}{(N-k)!}(N-j)^{N-k}$$

$$= (-1)^{N-1} + \frac{1}{(N-1)!}\sum_{k=2}^{N} s(N,k)(N-j)^{k-1} = (-1)^{N-1} - \frac{s(N,1)}{(N-1)!} = 0.$$

In the last step we have used (2.4).

Now let $l \neq j$. The coefficient of

$$P_{2l}P_{2J}, \ l + |J| = N, \ 1 \leq l \leq N-1$$

in (2.15) is given by

$$\frac{1}{(N-1)!} \sum_{k=1}^{N-1} \left( \sum_{i=0}^{k-1} s(N, N-i)(N-l)^{k-1-i} \right) (N-j)^{N-k} m_{(l,J)}^{(1)}$$

$$= \frac{1}{(N-1)!} \left( \sum_{2 \le i+k \le N} s(N, i+k)(N-l)^{i-1}(N-j)^k \right) m_{(l,J)}^{(1)}.$$

Lemma 2.1 implies that this sum vanishes.

Finally, we prove that the weight of $P_{2j}P_{2J}$ on the left-hand side of (2.9) coincides with its weight on the right-hand side. For this we write $J = (r, K)$ (with a possibly trivial $K$). For non-trivial $K$, we have $1 \le r < N - j$. In this case, (2.16) shows that the assertion is equivalent to

$$\sum_{k=1}^{N-j-1} \left( \frac{\sum_{i=0}^{k-1} s(N-j, N-j-i)(N-j-r)^{k-1-i}}{(N-j-1) \cdots (N-j-k+1)} \right) \frac{N^{N-j-k}}{(N-j-k)!}$$
$$= -m_{(j,r,K)}^{(1)}/m_{(r,K)}^{(1)} \binom{N-1}{j-1}^{-1}. \quad (2.17)$$

Using the abbreviation $M = N - j$, the left-hand side of (2.17) equals

$$\frac{1}{(M-1)!} \sum_{k=1}^{M-1} \sum_{i=0}^{k-1} s(M, M-i)(M-r)^{k-1-i} N^{M-k}$$
$$= \frac{1}{(M-1)!} \sum_{a+b \le M} s(M, a+b)(M-r)^{a-1} N^b.$$

We apply Lemma 2.1 to simplify this sum. We find

$$-\frac{1}{(N-j-1)!} \left( \frac{N(M-r) \cdots (-r+1) - (M-r)N(N-1) \cdots (N-M+1)}{(N-j-r)(j+r)} \right)$$
$$= \frac{1}{j+r} \frac{N!}{j!(N-j-1)!}.$$

The assertion (2.17) follows by combining this result with

$$m_{(j,r,K)}^{(1)}/m_{(r,K)}^{(1)} = -\frac{1}{j+r} \binom{N}{j}^2 \frac{j(N-j)}{N}.$$

For trivial $K$, i.e., $J = (r)$ and $r = N - j$, (2.16) shows that the assertion is equivalent to

$$\frac{1}{(r-1)!} \sum_{k=1}^{r-1} s(r, r-k+1)N^{r-k+1} + (-1)^{r-1} = -m_{(j,r)}^{(1)}/m_{(r)}^{(1)} \binom{N-1}{j-1}^{-1}. \quad (2.18)$$

Here, the term $(-1)^{r-1}$ on the left-hand side comes from the contribution of $\mathbf{C}_{2r}^{(r)}$. By (2.3) and (2.4), the left-hand side equals

$$\frac{1}{(r-1)!N} [N(N-1) \cdots (N-r+1) + (-1)^r(r-1)!N] + (-1)^{r-1} = \frac{(N-1)!}{j!(N-j-1)!}.$$

On the other hand,

$$m^{(1)}_{(j,r)}/m^{(1)}_{(r)} = -\frac{1}{N}\binom{N}{j}^2\frac{j(N-j)}{N}.$$

This yields (2.18). The proof is complete. $\hfill\square$

**Remark 2.1.** *Similar arguments can be used to prove the closed formula*

$$\pi_{2N}(\lambda) = \frac{1}{(N-1)!}\sum_{|I|=N}\frac{b_N(\lambda+\frac{n}{2}-N)}{\lambda+\frac{n}{2}-2N+I_l}m_I P_{2I} \qquad (2.19)$$

*(see (2.3)). Here $I_l$ denotes the most left entry of the composition $I$. Note that the coefficients in (2.19) are polynomials of degree $N-1$ since*

$$\lambda+\frac{n}{2}-2N+I_l = \left(\lambda+\frac{n}{2}-N\right)-(N-I_l)$$

*and the integers $N-I_l$ are zeros of $b_N(x)$. We omit the details.*

## 3. The recursive structure of $Q$-curvature

In the present section we prove Theorem 1.1.

The proof utilizes properties of $Q$-curvature polynomials. The notion of $Q$-curvature polynomials was introduced in [J09a] (see also [BJ10]). We briefly recall this concept. Assume that $n$ is even. Associated to any Riemannian manifold $(M, g)$ of dimension $n$, there is a finite sequence $Q_2^{res}(g;\lambda), Q_4^{res}(g;\lambda), \ldots, Q_n^{res}(g;\lambda)$ of polynomials of respective degrees $1, 2, \ldots, \frac{n}{2}$. These polynomials are defined by the constant terms

$$Q_{2N}^{res}(g;\lambda) = -(-1)^N D_{2N}^{res}(g;\lambda)(1) \qquad (3.1)$$

of the so-called residue families

$$D_{2N}^{res}(g;\lambda) : C^\infty([0,\varepsilon)\times M) \to C^\infty(M).^3$$

These are families of local operators which are defined in terms of the holographic coefficients $v_{2j}$ and the coefficients $\mathcal{T}_{2j}(\lambda)(f)$ in the asymptotic expansion

$$u \sim \sum_{j\geq 0} r^{\lambda+2j}\mathcal{T}_{2j}(\lambda)(f), \ \mathcal{T}_0(\lambda)(f) = f, \ r\to 0$$

of eigenfunctions

$$-\Delta_{g_+}u = \lambda(n-\lambda)u.$$

of the Laplace-Beltrami operator for the Poincaré-Einstein metric $g_+$ corresponding to $g$. The coefficients $\mathcal{T}_{2j}(g;\lambda)$ are meromorphic families (in $\lambda$) of differential operators on $M$. The residue families are conformally covariant generalizations of the GJMS-operators in the following sense. For any GJMS-operators $P_{2N}$, the family $D_{2N}^{res}(\lambda)$ contains $P_{2N}$ in the sense that

$$D_{2N}^{res}\left(g;-\frac{n}{2}+N\right) = P_{2N}(g)i^*, \qquad (3.2)$$

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[3] The name comes from their relation to a certain residue construction (see [J09a]).

where $i : M \hookrightarrow [0, \varepsilon) \times M$ denotes the embedding $m \mapsto (0, m)$. Moreover, $D_{2N}^{res}(g; \lambda)$ is conformally covariant in the sense that it satisfies the transformation law

$$e^{-(\lambda-2N)\varphi} D_{2N}^{res}(e^{2\varphi}g; \lambda) = D_{2N}^{res}(g; \lambda) \circ \kappa_* \circ \left(\frac{\kappa^*(r)}{r}\right)^{\lambda}$$

for all $\varphi \in C^{\infty}(M)$. Here, $\kappa$ denotes the diffeomorphism which relates the Poincaré-Einstein metrics of $g$ and $\hat{g} = e^{2\varphi}g$, i.e.,

$$\kappa^* \left(r^{-2}(dr^2 + g_r)\right) = r^{-2}(dr^2 + \hat{g}_r)$$

and $\kappa$ restricts to the identity on $r = 0$.

In terms of the families $\mathcal{T}_{2N}(\lambda)$ and the holographic coefficients, the $Q$-curvature polynomials are defined by

$$Q_{2N}^{res}(h; \lambda) = -2^{2N} N! \left(\left(\lambda + \frac{n}{2} - 2N + 1\right) \cdots \left(\lambda + \frac{n}{2} - N\right)\right)$$
$$\times \left[\mathcal{T}_{2N}^*(h; \lambda + n - 2N)(v_0) + \cdots + \mathcal{T}_0^*(h; \lambda + n - 2N)(v_{2N})\right].$$

Here, the overall polynomial factor has the effect to remove poles.

An additional important feature of residue families is that they satisfy a system of factorization identities which generalize (3.2). In fact, we have

$$D_{2N}^{res}\left(g; -\frac{n}{2} + 2N - j\right) = P_{2j}(g) D_{2N-2j}^{res}\left(g; -\frac{n}{2} + 2N - j\right), \; j = 1, \ldots, N. \quad (3.3)$$

Here, (3.2) is contained as the special case $j = N$; note that $D_0^{res}(g; \lambda) = i^*$. Now (3.3) implies that

$$Q_{2N}^{res}\left(-\frac{n}{2} + 2N - j\right) = (-1)^j P_{2j} Q_{2N-2j}^{res}\left(-\frac{n}{2} + 2N - j\right), \; j = 1, \ldots, N. \quad (3.4)$$

Note that $Q_0^{res}(\lambda) = -1$.

For full details on residue families and $Q$-curvature polynomials see [J09a] and [BJ10].

We continue with the

*Proof of Theorem 1.1.* The assertion is equivalent to

$$\sum_{a+|J|=N} m_{(J,a)}(-1)^a P_{2J}(Q_{2a}) = 2^{2N} N!(N-1)! w_{2N}, \; N \geq 1. \quad (3.5)$$

We prove (3.5) by comparing two different evaluations of the leading coefficient of the $Q$-curvature polynomial $Q_{2N}^{res}(\lambda)$. First, assume that $n$ is odd. On the one hand, Proposition 4.2 in [J09d] shows that the coefficient of $\lambda^N$ is

$$-2^{2N} N! w_{2N}. \quad (3.6)$$

On the other hand, the degree $N$ polynomial $Q_{2N}^{res}(\lambda)$ satisfies the identities

$$Q_{2N}^{res}\left(-\frac{n}{2} + 2N - j\right) = (-1)^j P_{2j} Q_{2N-2j}^{res}\left(-\frac{n}{2} + 2N - j\right), \; j = 1, \ldots, N-1 \quad (3.7)$$

and

$$Q_{2N}^{res}\left(-\frac{n}{2} + N\right) = -\left(\frac{n}{2} - N\right) Q_{2N} \quad (3.8)$$

(see (3.4)). Moreover, an analog of Theorem 1.6.6 in [BJ10] for odd $n$ states the vanishing result[4]

$$Q_{2N}^{res}(0) = 0.$$

These results show that (3.7) and (3.8) are equivalent to the identities

$$\mathcal{Q}_{2N}^{res}\left(-\frac{n}{2}+2N-j\right) = (-1)^j P_{2j} \mathcal{Q}_{2N-2j}^{res}\left(-\frac{n}{2}+2N-j\right), \; j = 1,\ldots,N-1 \quad (3.9)$$

and

$$\mathcal{Q}_{2N}^{res}\left(-\frac{n}{2}+N\right) = Q_{2N} \quad (3.10)$$

for the polynomials

$$\mathcal{Q}_{2N}^{res}(\lambda) = \lambda^{-1} Q_{2N}^{res}(\lambda). \quad (3.11)$$

By comparing the relations (3.9) and (3.10) with (2.9) and (2.10), Theorem 2.1 implies that the leading coefficient of $Q_{2N}^{res}(\lambda)$ equals

$$-\frac{1}{(N-1)!} \sum_{|J|+a=N} m_{(J,a)}(-1)^a P_{2J}(Q_{2a}). \quad (3.12)$$

Now the equality of (3.6) and (3.12) is equivalent to the asserted identity (3.5). Next, assume that $n$ is even. Then, under the additional assumption $n \geq 4N$, i.e., $-\frac{n}{2} + 2N \leq 0$, the sets

$$\left\{-\frac{n}{2}+2N-j \,|\, j = 1,\ldots,N\right\} \quad \text{and} \quad \{0\}$$

are disjoint, and the assertion follows by the same arguments as above. Thus, for fixed $N$, we have proved (3.5) in all dimensions $n \geq 4N$. Now we recall that all quantities in (3.5) are given by universal expressions in terms of the metric, its inverse, the curvature and covariant derivatives thereof with coefficients that are rational functions in $n$ which are regular for $n \geq 2N$. As a consequence, the relation (3.5) holds true also in the remaining cases $4N > n \geq 2N$ (for even $n$). $\qquad\square$

Theorem 1.1 is equivalent to

$$\mathcal{G}^2\left(\frac{r^2}{4}\right) = v(r). \quad (3.13)$$

This formulation naturally expresses the contributions of lower-order holographic co-efficients $v_{2j}$ ($2j < 2N$) on the right-hand side of (1.9) in terms of lower-order GJMS-operators acting on lower-order $Q$-curvatures. In fact, comparing coefficients in (3.13) yields the relations

$$2\Lambda_{2N} + \sum_{j=1}^{N-1} \frac{j(N-j)}{N}\binom{N}{j}^2 \Lambda_{2j}\Lambda_{2N-2j} = 2^{2N} N!(N-1)! v_{2N},$$

where

$$\Lambda_{2M} \overset{\text{def}}{=} \sum_{|a+|J|=M} m_{(J,a)}(-1)^a P_{2J}(Q_{2a}).$$

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[4]For odd $n$, the proof even simplifies since the families $\mathcal{T}_{2N}(\lambda)$ are regular at $n-2N$.

In particular, we find

$$(Q_4 + P_2(Q_2)) + Q_2^2 = 2!2^3 v_4$$

and

$$(Q_6 + 2P_2(Q_4) - 2P_4(Q_2) + 3P_2^2(Q_2)) + 6(Q_4 + P_2(Q_2))Q_2 = 2!3!2^5 v_6.$$

## 4. A SUMMATION FORMULA ON ROUND SPHERES

We recall that

$$(d/dt)|_0(e^{2Nt\varphi} Q_{2N}(e^{2t\varphi} g)) = (-1)^N P_{2N}^0(g)(\varphi), \tag{4.1}$$

where $P_{2N}^0$ denotes the non-constant part of $P_{2N}$, i.e., $P_{2N}^0 = P_{2N} - P_{2N}(1)$. For the proof of (4.1), we differentiate the identity

$$e^{(\frac{n}{2}+N)t\varphi} P_{2N}(e^{2t\varphi} g)(e^{-(\frac{n}{2}-N)t\varphi}) = P_{2N}(g)(1)$$

(see (1.1)) at $t = 0$. Using the decomposition

$$P_{2N} = P_{2N}^0 + (-1)^N \left(\frac{n}{2} - N\right) Q_{2N},$$

we find

$$-\left(\frac{n}{2} - N\right) P_{2N}^0(g)(\varphi) + (-1)^N \left(\frac{n}{2} - N\right) (d/dt)|_0(e^{2Nt\varphi} Q_{2N}(e^{2t\varphi} g)) = 0.$$

If $2N \neq n$, it suffices to divide this equation by $\frac{n}{2} - N$. In the critical case $2N = n$, (4.1) follows from the fundamental identity

$$e^{n\varphi} Q_n(e^{2\varphi} g) = Q_n(g) + (-1)^{\frac{n}{2}} P_n(g)(\varphi).$$

Now combining Theorem 1.1 with (4.1) implies a formula for the non-constant part of the self-adjoint operator

$$\mathcal{M}_{2N} \stackrel{\text{def}}{=} \mathbf{C}_{2N}^{(1)} = \sum_{|I|=N} m_I P_{2I}. \tag{4.2}$$

Conjecture 11.1 in [J09b] states that the operator $\mathcal{M}_{2N}$ actually can be identified with a certain *second-order* operator. In particular, a huge cancellation takes place. Since the sum defining $\mathcal{M}_{2N}$ contains the term $P_{2N}$ (we recall that $m_{(N)} = 1$), this relation can be seen as a recursive formula which expresses $P_{2N}$ in terms of lower-order GJMS-operators (and some additional terms). The following summation formula on round sphere is an important special case.

**Theorem 4.1** ([J09b]). *On the round sphere $\mathbb{S}^n$,*

$$\sum_{|I|=N} m_I P_{2I} = N!(N-1)!P_2, \ N \geq 1. \tag{4.3}$$

For the proof of an analog of Theorem 4.1 on the conformally flat pseudo-spheres we refer to [J09c].[5]

The following result provides an analogous summation formula in which the numbers $m_I = m_I^{(1)}$ are replaced by $m_I^{(2)}$.

**Theorem 4.2.** *On the round sphere* $\mathbb{S}^n$,

$$\sum_{|I|=N} m_I^{(2)} P_{2I} = -\frac{N!(N-1)!}{2!}(P_2^2 + NP_2), \ N \geq 1. \tag{4.4}$$

In particular, Theorem 4.2 shows that the operator of order $2N$ on the left-hand side is an operator of order four. We expect that for general metrics the operator on the left-hand side of (4.4) is an operator of order four, too.

*Proof of Theorem 4.2.* The arguments in the proof of Theorem 4.1 in [J09b] show that

$$\sum_{a+|J|=N} m_{(a,J)}^{(1)} a P_{2a} P_{2J}$$
$$= \sum_{s=0}^{N-1} (-1)^{N+s} P_{2(N-s)} \frac{N!(N-1)!^2}{(N-s)!s!(N-s-1)!^2} \sum_{a=1}^{N} (-1)^a a \binom{N-s-1}{a-1};$$

note that the right-hand side of this formula differs from the last formula in the proof of Lemma 6.3 in [J09b] only by the additional coefficient $a$ in the sum over $a$. Now the summation formula

$$\sum_{a \geq 0} (-1)^a (a+1) \binom{n}{a} = \begin{cases} 0 & n \geq 2 \\ -1 & n = 1 \\ 1 & n = 0 \end{cases}$$

implies that, in the above sum, the sum over $a$ vanishes except for $s = N - 1$ and $s = N - 2$. These two contributions yield

$$N!(N-1)!P_2 + \frac{N-1}{2}N!(N-1)!P_4 = N!(N-1)!\left(\frac{N-1}{2}P_2^2 + NP_2\right)$$

by using $P_4 = P_2(P_2 + 2)$. Now Theorem 4.1 and

$$m_{(a,J)}^{(2)} = \left(\frac{N-a}{N-1} - \frac{N}{2}\right) m_{(a,J)}^{(1)}$$

(see (2.5)) show that

$$\sum_{a+|J|=N} m_{(a,J)}^{(2)} P_{2a} P_{2J}$$

equals

$$N!(N-1)!\left(\left(\frac{N}{N-1} - \frac{N}{2}\right)P_2 - \frac{1}{N-1}\left(\frac{N-1}{2}P_2^2 + NP_2\right)\right).$$

Simplification yields the assertion.                                                    $\square$

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[5]A full proof of Conjecture 11.1 will appear in [J11a].

**Remark 4.1.** *Similar arguments prove the summation formulas*

$$\sum_{|I|=N} m_I^{(3)} P_{2I} = \frac{N!(N-1)!}{3!2!}(P_2^3 + (3N-1)P_2^2 + N(3N-1)/2P_2),\ N \geq 1$$

*on $\mathbb{S}^n$.*

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